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Localisation of fermions to brane: codimension $d \geq 2$

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Abstract: We investigate $4 + d$ dimensional fermionic models in which the system in codimension-d supports a topologically stable solution, and in which the fermion may be localised to the brane, with power law in 'instanton' backgrounds and exponentially in 'soliton' backgrounds. When the fermions are isoscalars, the mechanism fails, while for isospinor fermions it is successful. As backgrounds we consider instantons of Yang-Mills and sigma models in even codimensions, solitons of sigma models in odd codimensions, as well as solitons of Higgs and Goldstone models in all codimensions.

Keywords: Field Theories in Higher Dimensions, Solitons Monopoles and Instantons, Extra Large Dimensions.

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Contents

1. Introduction

The idea that our world is a (fairly flat) 3-brane in a higher dimensional space has deep roots in the 19th century [1]. It was introduced into physics in the context of cosmological defects on one hand and the branes of M-theory on the other. In any case one has to explain why the observed fermion masses are so much smaller than the mass scale given by the transversal extension of the brane. On some parameter range one expects the branes to approach classical configurations, such that a semiclassical description of matter on the

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brane is possible. Qualitative features like the appearance of low energy excitations on the brane should be independent of the parameters at least locally, such that these semiclassical descriptions may be relevant for realistic cases.

We study the extension of the original work of Rubakov and Shaposhnikov [2] localising a fermion to the brane, in which a 5 dimensional model, i.e. one with codimension-1 is employed. There [2, 3], the brane lagrangian is the ϕ^4 (or sine-Gordon) system in 1 dimension, supporting the kink-soliton. We are concerned with the possibilty of extending this mechanism to the case of arbitrary codimension-$d \geq 2$. In general, fermions are localised when their reduced Dirac equation in the d dimensions transversal to the brane has zero modes. This always is the case when the index of the corresponding Dirac operator is positive.

In case [2] the wave function of the fermion drops exponentially away from the brane. Such backgrounds will be called solitonic. For them an explicit formula for the index is known [4], and physically relevant specialisations are known both in odd and in even dimension [5, 6]. The index only depends on the behaviour of the scalar fields at infinity, gauge fields are assumed to approach zero at infinity and do not influence the index in a direct way. Nevertheless, the corresponding internal symmetries are crucial. In the backgrounds of Yang-Mills or sigma model instantons, the fermionic fields have a power-law fall-off away from the brane. In such cases the available index calculations depend on compactifications at infinity, which allow an application of the Atiyah-Singer index theorem [7].

For solitonic backgrounds, non-zero indices occur in both even and odd dimensions, whereas in the instanton case one needs even dimensions. Even when the index is zero or negative, solutions of the reduced Dirac equation might exist. For fermions in instanton backgrounds this can be excluded by vanishing theorems for the kernel of the corresponding Dirac operator (the Weizenböck formula), but in other contexts little is known. In particular, no general results for solitonic backgrounds seem to exist.

The index formulas require cumbersome if straightforward integrations, but often an analytic approach is possible. When a background can be deformed into a superposition of well separated configurations with spherical symmetry, the index is additive and only these spherically symmetric situations have to be considered. For any of them the index is the sum over the indices for fixed angular momentum, which are determined by the behaviour of ordinary differential equations and easy to calculate. We consider some families of such configurations. As new features we obtain that certain solutions are universal and apply to any number of transversal dimensions, and in certain cases a vanishing theorem can be established.

We have considered two alternative types of fermionic models in $4 + d$ dimensions. In the first of these, (i), the Dirac spinor is *isoscalar*, such that the Dirac operator on the codimension involves only unmodified partial derivatives and no gauge field. In the second type, (ii), the Dirac spinor on the codimension is in general an *isospinor* under an internal SO(d) group, i.e. it is a square matrix valued array. The Dirac operator features a covariant derivative, but the index is determined for the case of a vanishing gauge field. Here we also consider an exceptional $d = 2$ case when the covariant derivative is abelian and the Dirac spinor is a two-component column.

Like in [2, 3], our brane lagrangians in d dimensions support finite energy topologically stable solutions, both of 'instanton' and of 'soliton' types. Our nomenclature here is the following: consider the topological charge density $\varrho[\varphi]$, e.g. the Chern-Pontryagin (C-P) density and its descendents when $[\varphi]$ symbolises YM fields (in even dimensions) and YM-Higgs fields, respectively, or, the degree of the map when $[\varphi]$ stands for sigma model fields. The scalar deansity $\varrho[\varphi]$ is *essentially total divergence*, in the sense that its variation vanishes when it is subjected to arbitrary variations $\delta\varphi$. (Indeed in the (C-P) case $\varrho[\varphi]$ is precisely a *total divergence* $\varrho[\varphi] = \partial_i\Omega_i$, Ω_i being the Chern-Simons density.) When $\varrho[\varphi]$ is subjected to spherical symmetry, which is the case for the fields $[\varphi]$ at infinity, it reduces always to a *total derivative* of some function σ, i.e.

$$\int \varrho \; d^d x \propto \int \frac{d\sigma}{dr} dr \, . \tag{1.1}$$

We denote this function pertaining to the classical solutions φ_c by $\sigma_c = \sigma[\varphi_c]$. The profiles (2.13) and (2.14) stated below in section 2.1 specify our nomenclature of 'instanton' and 'soliton' types, respectively. Thus, finite energy topolologically stable solutions to Yang-Mills (YM) or a sigma model systems in even d are typified by 'instanton' profiles, (2.13), and solutions to sigma models in odd d, or to Higgs models, i.e. YM-Higgs (YMH) systems, or Goldstone models, in both even and odd d, by 'soliton' profiles (2.14). By Goldstone model, in turn, we mean the gauge decoupled version of a Higgs model, provided of course that the soliton persists in the gauge decoupling limit of the YMH model in question. In this context, the solutions of the symmetry breaking model(s) employed in [2, 3] are typified by 'soliton', (2.14), profiles.

The main feature of the mechanism of [2, 3] is dimensional descent of a $4+d$ dimensional fermionic model to 4 dimensional Minkowski space, the Dirac field on which is assumed to be chiral and to be massless. For codimension-$d \geq 2$, after the descent one is left with a nontrivial Dirac equation on the codimension-d, which we have called the *residual Dirac equation*. The crucial step is that of finding normalisable zero modes of this residual Dirac equation. The asymptotic behaviour of the residual Dirac spinor is then responsible for the decay of the Dirac field off the brane. Whether this decay is achieved, and if so, is it power like or exponential, will depend on the $4+d$ dimensional fermionic model chosen.

It will turn out that the desired normalisable zero modes do not exist for models of type (i) for any codimension d, and that such solutions exist for models of type (ii) for all d.

To date, extension to higher codimensions ($d \geq 2$) for this mechanism has been performed for codimension-2, in a series of works started by Libanov and Troitsky [8, 9] in flat space, and in the presence of gravity, in [10]. The models employed in both [8] and [10] are of the type (ii), namely featuring a covariant derivative in the Dirac operator. In these models [8, 10], what guarantees the existence of zero modes of the residual Dirac equation in codimension-2, is the choice of *such* a Yukawa coupling in the 6 dimensional model *that* leads to a residual Dirac equation which coincides with the particular Dirac equation on an abelian-Higgs background for which Jackiw and Rossi [11] have constructed the zero modes explicitly. To follow this line of approach for codimensions-$d \geq 3$, one is naturally led to considering such $4+d$ dimensional fermionic models which result in d dimensional residual Dirac equations for which we know there exist normalisable zero modes, or better still that we can construct such solutions. Dirac equations in d dimensions in the background of a YM 'instanton' or a YMH 'soliton' (or its associated Goldstone 'soliton') supporting such zero modes are the natural candidates which will be proposed. Dirac equations in d dimensions will be solved in the appropriate 'instanton' or 'soliton' backgrounds, supported respectively by a hierarchy of YM and YMH models in these dimensions.

The first in the hierarchy of YM models is the usual YM system in 4 dimensions, and its extensions to all even dimensions as given in [12] support 'instantons' in these dimensions, analogous to the 4 dimensional instanton [13]. It is these 'instantons' in $d = 2n$ dimensions that we will employ as backgrounds for the construction of the zero modes of the residual Dirac equations, extending[1] the $d = 4$ result of Jackiw and Rebbi [14] to arbitrary d.

The first in the hierarchy of non abelian YMH models is the Georgi-Glashow (Higgs) model in $d = 3$, which supports the the usual monopole [16]. The zero modes of the Dirac equation on this background was given long ago by Jackiw and Rebbi [17]. The extension to arbitrary d is completely straightforward: The result of [17] follows directly from the existence of monopoles [16] in the 3 dimensional Higgs model. The corresponding solitons of Higgs models in arbitrary dimensions d have been systematically shown to exist, being constructed numerically in [18]–[20]. The generalisation of the $d = 3$ result of [17] to arbitary d then follows almost trivially. Moreover, the adaptation of the result of [17] to the case of Goldstone model backgrounds also follows systematically, by employing the solitons presented in [21]–[23].

In section 2 we present the $4+d$ dimensional models on the space with coordinates $x_M = (x_\mu, x_m)$, $\mu = 0, \ldots, 3$ labeling the Minkowski space and $m = 1, 2, \ldots, d$ the codimension, and, we give the Ansatz separating the varaibles x_μ and x_m in the field equations. This describes the dimensional descent. Both type (i) and type (ii) models, i.e. with Dirac operators featuring both *partial* derivatives and *covariant* derivatives, are presented in this section. The resulting residual Dirac equations in d dimensions will then be examined in detail in the subsequent sections 3 and 4 respectively. In 3, it will be shown that for models of type (i) the fermion cannot be localised to the brane for any d. In section 4 type (ii) models will be analysed. In the subsection 4.1, zero modes of the residual Dirac equations in even $d \geq 4$ dimensional 'instanton' backgrounds of YM systems will be constructed, resulting in the *power localisation* of the fermion to the brane. In the subsection 4.2, the corresponding zero modes in all $d \geq 2$ dimensional 'soliton' backgrounds of Higgs (or their associated Goldstone) systems will be constructed, resulting in the *exponential localisation* of the fermion to the brane. A summary of the results is given in section 5, and three appendices have been supplied. Appndix A describes the (even) $d = 2n$ dimensional YM models and their 'instantons'. Appendix B describes the d dimensional Higgs models and their 'solitons', and appendix C describes the Goldstone counterparts of the latter.

[1] We restrict our considerations to Dirac equations on spherically symmetric backgrounds only and exclude the multicentre backgrounds employed in [14], or even periodic backgrounds used in [15].

2. The model(s) and residual Dirac equations

We will consider the following two types of fermionic actions, formally expressed as

$$S_\Psi = \int d^4x \; d^dx \; \left(\bar{\hat{\Psi}}\hat{\Gamma}^M\partial_M\hat{\Psi} - \mu\,\sigma[\varphi]\;\bar{\hat{\Psi}}\hat{\Psi}\right) \tag{2.1}$$

$$S_\Psi = \int d^4x \; d^dx \; \left(\bar{\hat{\Psi}}\,\hat{\Gamma}^M D_M\hat{\Psi} - \mu\,\bar{\hat{\Psi}}\,\Xi[\varphi]\hat{\Psi}\right) \tag{2.2}$$

The first of these, (2.1), pertains to the type (i) family of models featuring partial derivatives in the codimension. The components of the spinor field on the codimension-d, in (2.1), are *isoscalar*. $\sigma[\varphi]$ in (2.1) is a scalar function of the fields $[\varphi]$ symbolising the scalar and/or the YM field describing the brane lagrangian in the codimension-d. Specifically, it will be defined as the leading term in the spherically symmetric restriction of the *topological current*, e.g. the Chern-Simons term in the case of YM. Both 'instanton' and 'soliton' backgrounds can be accommodated in this scheme.

The second, (2.2), represents type (ii) models in which $D_M = (\partial_\mu, D_m)$. The components of the spinor field on the codimension-d, in (2.2), are *isospinor* except in the case $d = 2$ case when they are *isoscalar*. These models subdivide further into two subclasses, ones with $\mu = 0$, i.e. without a Yukawa term, and, those with $\mu \neq 0$. The models (2.2) with $\mu = 0$ accommodate only 'instanton' backgrounds, while those with $\mu \neq 0$ accommodate only 'soliton' backgrounds, and in this case $\Xi[\varphi]$ is a matrix valued function of $[\varphi]$.

The Dirac equations arising from (2.1) and (2.2) are, respectively

$$\hat{\Gamma}^M\partial_M\hat{\Psi} + \mu\;\sigma[\varphi_c]\,\hat{\Psi} = 0 \tag{2.3}$$

$$\hat{\Gamma}^M D_M\hat{\Psi} + \mu\;\Xi[\varphi_c]\,\hat{\Psi} = 0\,, \tag{2.4}$$

to be solved on the classical background $\varphi_c(r)$ to be precised later. In the present work, we anticipate the use of radially symmetric background solutions in terms of the radial variable $r = |x_m|$ of the co-dimension, though a richer spectrum of such backgrounds arises when this symmetry is relaxed.

Denoting the 4-dimensional (spacetime) coordinates by x_μ and the coordinates of the codimension by x_m, we represent the $4+d$ dimensional gamma matrices $\hat{\Gamma}_M = (\hat{\Gamma}_\mu, \hat{\Gamma}_m)$ by

$$\hat{\Gamma}_\mu = \gamma_\mu \otimes \mathbb{1}\,, \qquad \hat{\Gamma}_m = \gamma_5 \otimes \Gamma_m \tag{2.5}$$

in terms of the 4-dimensional gamma matrices γ_μ and their chiral matrix γ_5, and the d-dimensional gamma matrices Γ_m.

Our separability Ansatz, which also effects the dimensional descent, is

$$\hat{\Psi}(x_\mu, x_m) = \Psi(x_\mu) \otimes \psi(x_m)\,, \tag{2.6}$$

and when applicable,

$$\Xi[\varphi(x_m)] = \mathbb{1} \otimes \xi[\varphi(x_m)]\,. \tag{2.7}$$

In (2.7), the array $\xi[\varphi(x_m)]$ is a matrix valued array whose size will be determined by the representaion in which the gauge connection in the covariant derivative D_m is. (In the generic case this will be the Gamma matrix representation of SO(d).) Detailed definitions of ξ for particular models to be considered, are given in section 2.2.

Using (2.5) and the separability Ansatz (2.6),(2.7), the Dirac equations (2.3) and (2.4) yield, respectively

$$\gamma^{\mu}\partial_{\mu}\Psi\otimes\psi+\gamma^5\Psi\otimes\Gamma^m\partial_m\psi+\mu\ \sigma_c\ \Psi\otimes\psi\ =0 \tag{2.8}$$

$$\gamma^{\mu}\partial_{\mu}\Psi\otimes\psi+\gamma^5\Psi\otimes\Gamma^m D_m\psi+\mu\ \Psi\otimes\xi_c\ \psi\ =0\,, \tag{2.9}$$

in which we have used the notation $\sigma_c = \sigma[\varphi_c]$ and $\xi_c = \xi[\varphi_c]$. If we now invoke the existence of the zero modes of the Dirac field in 4 dimensional spacetime

$$\gamma^{\mu}\partial_{\mu}\Psi=0\,,$$

and require that the Dirac spinor is chiral, i.e. that

$$\gamma^5\Psi=\Psi\,,$$

then (2.8),(2.9) finally reduce to the *residual* Dirac equations in d dimensions

$$(\Gamma^m\partial_m+\mu\ \sigma_c)\ \psi\ =0 \tag{2.10}$$

$$(\Gamma^m D_m+\mu\ \xi_c)\ \psi\ =0\,. \tag{2.11}$$

It is in order to mention at this point, that for the case of $\mu \neq 0$ models (2.2), the covariant derivative D_m in (2.11) will eventually be replaced by the partial derivative ∂_m, since the detailed analysis of (2.11) to be carried out subsequently is restricted only to Goldstone models associated to the Higgs models described in appendix C, namely to the gauge decoupled versions of the associated Higgs models described in the appendix B. We stress that all our results are valid also for the Higgs model 'soliton' backgrounds, and the only reason we eschew working with the latter is that the corresponding analysis of (2.11) yields qualitatively the same results as in the Goldstone case, and the analysis in the latter case is somewhat simpler.

We will seek solutions to (2.10),(2.11) satisfying

$$\psi'(0)<\infty\,,\qquad ||\psi||<\infty\,. \tag{2.12}$$

It will turn out that there exist no solutions of (2.10) satisfying (2.12) for any d, but will find such solutions to (2.11) for all d.

2.1 Definitions of $\sigma[\varphi]$

$\sigma[\varphi]$ are defined as the leading terms in the *topological currents* at infinity, i.e. when the fields defining them are spherically symmetric. This subsection is subdivided in two parts.

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In the first, we give the definition of $\sigma[\varphi]$ for a topologically stable background supported by a $O(d+1)$ sigma model, as well as a background supported by a $\mathrm{SO}(d=2n)$ YM system in even, $d=2n$, dimensions. The latter are the spherical-symmetrically restricted Chern-Simons densities of $\mathrm{SO}(d=2n)$ YM field, which turn out to be described by essentially the same 'instanton' type profile

$$-1 \underset{r\leftarrow 0}{\longleftarrow} \sigma_c \underset{r\rightarrow\infty}{\longrightarrow} 1\,, \tag{2.13}$$

as in the case of *even* dimensional sigma models.

In the second we consider backgrounds supported by Higgs or Goldstone models, typified by 'soliton' profiles

$$0 \underset{r\leftarrow 0}{\longleftarrow} \sigma_c \underset{r\rightarrow\infty}{\longrightarrow} 1\,, \tag{2.14}$$

as in the case of *odd* dimensional sigma models. Detailed expositions of (2.13) and (2.14) are given in the following two subsections.

2.1.1 $\sigma[\varphi]$ for $O(d+1)$ sigma model and $\mathrm{SO}(d=2n)$ pure YM backgrounds

d dimensional $O(d+1)$ sigma models and their topologically stable solitons have been discussed extensively elsewhere [24] so we do not elaborate on them here. Best known amongst these is the $d=2$ dimenesional *scale invariant* $O(3)$ sigma model whose solitons, namely the well known Belavin-Polyakov vortices [25], are evaluated in closed form. Here we are concerned only with the topological boundary conditions the relevant solitons satisfy. Moreover, as noted above, we will restrict to the case of radial (spherically symmetric) solitons. So we state these, in terms of the $d+1$ component scalar fields $\chi^a=(\chi^m,\chi^4)$, subject to the constraint $|\chi^a|^2=1$:

$$\chi^m = \sin f(r)\ \hat{x}^m\,, \qquad \chi^{d+1}=\cos f(r)\,. \tag{2.15}$$

In (2.15) $\hat{x}^m = r^{-1}x^m$ is the unit vector in the codimension. The topological charges stabilising the solitons of these models are the winding numbers, which take on unit values provided that the solutions satisfy the asymptotic conditions[2]

$$\lim_{r\to 0} f(r)=0\,, \qquad \lim_{r\to\infty} f(r)=\pi\,. \tag{2.16}$$

Our definition of the function $\sigma_c=\sigma[f(r)]$ corresponding to the solution $\varphi_c=f(r)$ of this model, is that given by (1.1).

We list the functions $\sigma_c(d)$ for this model, for the cases $d=2,\ d=4$ and $d=3,\ 5$, separately for even and odd d. Up to unimportant multiplicative constant depending on the angular volumes, we find following [24],

$$\sigma(d=2)\propto\ \cos f\,, \qquad \sigma(d=4)\propto\ \left(\cos f-\frac{1}{3}\cos^3 f\right), \quad \dots \tag{2.17}$$

[2]The more usual alternative $\lim_{r\to 0} f(r)=\pi$, $\lim_{r\to\infty} f(r)=0$ is not adopted, for the sake of making contact with the usual asymptotics (2.22) for the corresponding YM fields.

for **even** $d = 2, 4$ respectively, and

$$\sigma(d=3) \propto\ f - \frac{1}{2}\sin 2f \quad , \quad \sigma(d=5) \propto\ \left(\frac{3}{2}f - \sin 2f + \frac{1}{4}\sin 4f\right) \ , \quad \dots \tag{2.18}$$

for **odd** $d = 3, 5$ respectively. We see that the ranges of these topological charge densities are quite different for even and odd d.

(2.17) and (2.18) result, qualitatively, in the following profiles of σ_c

$$+1 \underset{r\leftarrow 0}{\longleftarrow} \sigma_c \underset{r\to\infty}{\longrightarrow} -1 \qquad , \qquad \text{for even} \quad d \tag{2.19}$$

$$0 \underset{r\leftarrow 0}{\longleftarrow} \sigma_c \underset{r\to\infty}{\longrightarrow} 1 \qquad , \qquad \text{for odd} \quad d \ . \tag{2.20}$$

We next turn to the SO(d) YM system in *even* d-diemensions, for which the spherically symmetric Ansatz, analogous to (2.15), is

$$A_m = \frac{1-w(r)}{r}\,\Sigma_{mn}^{(\pm)}\hat{x}_n\,, \qquad \Sigma_{mn}^{(\pm)} = -\frac{1}{4}\left(\frac{1\pm\Gamma_{d+1}}{2}\right)[\Gamma_m,\Gamma_n]\,. \tag{2.21}$$

Analogously to (2.16), the 'instanton' boundary conditions that result in topologically stable (anti)-selfdual solutions to the systems of YM hierarchies [12], are

$$\lim_{r\to 0} w(r) = +1 \quad , \quad \lim_{r\to\infty} w(r) = -1\,, \tag{2.22}$$

which coincides with (2.16) under the replacement

$$\cos f(r) \longleftrightarrow w(r)\,, \quad \dots \tag{2.23}$$

Now in all *even* diemensions there exist Chern-Pontryagin charge densities, whose spherically symmetric restrictions are the analogues of (2.17). Up to unimportant numerical factors, these densities in dimensions $d = 4$ and $d = 6$ are

$$\sigma(d=4) \propto\ \left(w - \frac{1}{3}w^3\right), \qquad \sigma(d=6) \propto\ \left(w - \frac{2}{3}w^3 + \frac{1}{5}w^5\right). \tag{2.24}$$

Note that the first ($d = 4$) member of (2.24) coincides with the second ($d = 4$) member of (2.17) under the replacement (2.23). This is a recurring coincidence. It is obvious now that the profile of σ_c in this case coincides with (2.19).

2.1.2 $\sigma[\varphi]$ for SO(d) Higgs/Goldstone model backgrounds

We now consider topologically stable backgrounds supported by d dimensional SO(d) Higgs models [18]–[20] and their associated Goldstone models [21]–[23].

The topologiocal charges of these models are described by scalar fields ϕ^m , $m = 1, 2, \dots, d$, in d dimensions. Apart from the various kinetic terms, Goldstone models are distinguished by a symmetry breaking self-interaction potential, leading to the important asymptotic condition

$$\lim_{r\to\infty} |\phi^m|^2 = \eta^2 \tag{2.25}$$

in which η is the VEV with inverse dimensions of length. Here again, we restrict to the radially symmetric fields

$$\phi^m = \eta\, h(r)\, \hat{x}^m\,, \tag{2.26}$$

and for the special case of $d = 2$ dimensions, the radially symmetric *vorticity* n field is

$$\phi^m = \eta\, h(r)\, n^m\,, \qquad n^m = (\cos n\phi, \sin n\phi)\,. \tag{2.27}$$

The topological charges stabilising the solitons of these models are the winding numbers, which take on the unit value for the following asymptotic conditions

$$\lim_{r\to 0} h(r) = 0\,, \qquad \lim_{r\to\infty} h(r) = 1\,. \tag{2.28}$$

The function σ_c is now expressed in terms of the classical soliton profile $h(r)$. One difference from the sigma models of the previous subsection however is, that the $d = 1$ case for Goldstone models, unlike the sigma models, does not trivialise but coincides, for example, with the φ^4 model. Another difference is that the winding number density for the radial fields (2.26) does not take qualitatively different expressions for even and odd d, as in (2.17)–(2.18).

In both even and odd d-dimensional Higgs and Goldstone models, the leading term in the winding number density turns out to be proportional to

$$h(r)^d \qquad \Rightarrow \qquad \sigma_c \stackrel{\rm def}{=} \eta^{-d}|\phi^m|^d \equiv \eta^{-d}\phi^d \tag{2.29}$$

and since in the following we will need only the asymptotic values and not detailed behaviours of $\sigma_c(r)$, we omit the d-th power of $h(r)$ in (2.29) and simply state the topologically meaningful asymptotic behaviour

$$0 \stackrel{r\leftarrow 0}{\longleftarrow} \sigma_c \stackrel{r\rightarrow\infty}{\longrightarrow} 1\,, \qquad \text{for all} \quad d\,. \tag{2.30}$$

2.2 Definitions of $\Xi[\varphi]$

Unlike the quantities $\sigma[\varphi]$ presented in the previous subsection, which are isoscalar, the quantities $\xi[\varphi]$ in (2.7) are matrices with isotopic indices. In models employing sigma model or YM 'instanton' backgrounds in even codimension-d, $\mu = 0$ so that $\Xi[\varphi]$ is defined only for models employing 'soliton' backgrounds with $\mu \neq 0$. With odd d sigma model backgrounds in turn, there is no useful efinition for $\xi[\varphi]$. The reason is simple, and hinges on the requirement that the residual Dirac equation (2.11), like (2.10), should develop a mass term asymptotically in the codimension.

Let us examine the Yukawa term in (2.11) in the asymptotic region, which is subject to spherical symmetry. Consider the matrix valued function $\xi[\varphi]$ in (2.7) in terms of the two alternative codimension fields $\chi^a = (\chi^m, \chi^{d+1})$, pertaining to the $O(d+1)$ sigma model, and ϕ^m, to the Higgs or Goldstone model. The only natural forms for matrix valued ξ in the gamma matrix represenation in isospace are proportional to the following matrix valued quantities

$$\xi \propto \Gamma_m\, \chi^m\,, \qquad \xi \propto \Gamma_m\, \phi^m\,, \tag{2.31}$$

respectively. Inspecting the spherically symmetric Ansätze (2.15), (2.26) and the asymptotics (2.16), (2.28) required, one sees that the only Yukawa term which leads to a nonvanishing mass term is the second member of (2.31). Henceforth, models of type (2.2) with $\mu > 0$ will be restricted to Higgs or Goldstone model 'soliton' type backgrounds only, with the corresponding Yukawa term determined by the second member of (2.31). The type (ii) models considered are typified by the definitions of the quantity ξ. There will be two such choices.

The first applies in the case where the gauge connection in the covariant derivative is abelian, which is the case only for $d = 2$, e.g. in the background of the usual abelian Higgs model. In this case, it is possible to take the Dirac field to be an *isoscalar*, and our choice for ξ is

$$\xi = \sigma_1 \sigma_m \phi^m = \phi^1 \mathbb{1} + i\phi^2 \sigma_3 \,, \qquad m = 1, 2 \,, \tag{2.32}$$

which is (Euclidean) Lorentz invariant.

The second concerns the case of generic codimension-d, where the SO(d) connection is non abelian, our choice for ξ is

$$\xi = \mathbb{1} \otimes \Gamma_m \phi^m \,, \tag{2.33}$$

where the matrix $\mathbb{1}$ is labeled by the spinor indices and the matrix $\Gamma_m \phi^m$ is labeled by the isospinor indices. While (2.33) is defined for non abelian backgrounds, i.e. for $d \geq 3$, it applies also to the $d = 2$ case formally. In that case, we express the abelian gauge connection, say a_m, in formally antihemitean form

$$A_m = \frac{i}{2}\, a_m\, \sigma_3 \tag{2.34}$$

acting on the matrix valued Higgs field $\Phi = \phi^m \sigma_m$. The covariant derivative in (2.11) for the $d = 2$ case of the generic model is then defined by the connection (2.34).

3. Type (i) models with isoscalar $\psi(x_m)$

We will show that type (i) models with action (2.1), on backgrounds with either type of profile (2.13) and (2.14) of the function σ_c, do not support solutions satisfying condition (2.12), for any codimension d. This section is divided into four subsections. In 3.1–3.3 we analyse the residual Dirac equation (2.10) for $d = 2$, $d = 3$ and arbitrary d respectively. This yields a set of coupled ordinary differential equations, which are of the same form for all d. Then in 3.4 we show that these equations do not have solutions satisfying the required property (2.12).

3.1 Codimension $d = 2$

The 2 component residual spinor $\psi(x_m)$ is subjeted to radial symmetry

$$\psi = \begin{pmatrix} f_1\ e^{im\phi} \\ f_2\ e^{im'\phi} \end{pmatrix}, \tag{3.1}$$

with m and m', both integers. Denoting m, m' instead by l, l', for uniformity of notation for all d, the variables r and ϕ in equation (2.10) separate for $l' = l + 1$, resulting in the pair of coupled first order equations

$$f_1' - \frac{l}{r} f_1 + \sigma_c f_2 = 0 \tag{3.2a}$$

$$f_2' + \frac{l+1}{r} f_2 + \sigma_c f_1 = 0 \,. \tag{3.2b}$$

3.2 Codimension $d = 3$

The residual 2 component spinor $\psi(x_m)$ transforms as a spin-$\frac{1}{2}$ spinor under 3 dimensional rotations, and to achieve a separation of variables we employ the spinor harmonics [26] $\Omega^{(\pm)}_{lm}$ to expand ψ

$$\psi = f_1(r)\, \Omega^{(+)}_{lm} + f_2(r)\, \Omega^{(-)}_{l'm} \,. \tag{3.3}$$

The spinor harmonics are defined as

$$\Omega^{(\pm)}_{lm} = C(l,1,l\pm 1)^{m-\frac{1}{2},+\frac{1}{2}}_{\;m} Y_{l,m-\frac{1}{2}}(\theta,\phi)\chi_{+\frac{1}{2}} + C(l,1,l\pm 1)^{m+\frac{1}{2},-\frac{1}{2}}_{\;m} Y_{l,m+\frac{1}{2}}(\theta,\phi)\chi_{-\frac{1}{2}} \tag{3.4}$$

in which $Y_{l,m}(\theta,\phi)$ are the spherical harmonics and $\chi_{\pm\frac{1}{2}}$ are the constant valued 2 component eigenvectors for spin-$\frac{1}{2}$

$$\chi_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{3.5}$$

Evaluating the Clebsch-Gordan coefficients in (3.4) and substituting (3.5), we have

$$\Omega^{(+)}_{lm} = \begin{pmatrix} \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}\, Y_{l,m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}\, Y_{l,m+\frac{1}{2}}(\theta,\phi) \end{pmatrix}, \qquad \Omega^{(-)}_{lm} = \begin{pmatrix} -\sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}\, Y_{l,m+\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}\, Y_{l,m-\frac{1}{2}}(\theta,\phi) \,. \end{pmatrix} \tag{3.6}$$

The result of acting with the gradient operator on the spherical harmonics can be systematically calculated applying the Clebsch-Gordan series [26]. Applying this to the residual Dirac equation (2.10) with the Ansatz (3.3), and setting $l' = l + 1$, we have

$$\left(f_1' - \frac{l}{r}\, f_1 + \sigma_c\, f_2 \right) \Omega^{(-)}_{l+1,m} + \left(f_2' + \frac{l+2}{r}\, f_2 + \sigma_c\, f_1 \right) \Omega^{(+)}_{l,m} = 0 \tag{3.7}$$

resulting in

$$f_1' - \frac{l}{r} f_1 = -\sigma_c f_2 \tag{3.8a}$$

$$f_2' + \frac{l+2}{r} f_2 = -\sigma_c f_1 \,. \tag{3.8b}$$

The similarity of (3.8a)–(3.8b) with (3.2a)–(3.2b) is manifest and holds for arbitrary codimension, as we shall see immediately below.

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3.3 Arbitrary codimension d

To generalise to arbitrary codimension d, we need some preparation. Let $p_k = -i\partial_k$, $\sigma_{ij} = i[\Gamma_i, \Gamma_j]/2$, $i, j = 1, \dots d$, and

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij} (x_i p_j - x_j p_i) \, .$$

Squaring yields $\mathcal{A}^2 = (n-2)\mathcal{A} + L^2$, where

$$L^2 = \sum_{i<j} (x_i p_j - x_j p_i)^2 \, .$$

It is well known that the eigenvalues of L^2 are $l(l+d-2)$, $l = 0, 1, \dots$. The corresponding possible eigenvalues of $\mathcal{A}$ are $-l$ and $l+d-2$, but by inspection only 0 for $l = 0$. Thus the list of $\mathcal{A}$ eigenvalues is $0, -1, -2, \dots$ plus $d-1, d, d+1 \dots$.

In analogy to the case $d = 3$ we introduce the spinor harmonics $\Omega^{(+)}_{lM}$, $\Omega^{(-)}_{lM}$, where

$$\begin{aligned} L^2 \Omega^{(\pm)}_{lM} &= l(l+d-2) \Omega^{(\pm)}_{lM} \, , \\ \mathcal{A} \Omega^{(+)}_{lM} &= -l \Omega^{(+)}_{lM} \, . \end{aligned}$$

The index M stands collectively for eigenvalues of angular momentum operators which commute with $\mathcal{A}$. Determining the spinor harmonics explicitly is more complicated than for $d = 3$, but not necessary for our purpose.

The operator $\Gamma^m \hat{x}_m$ transforms eigenspaces for $-l$ and $(d-1+l)$ into each other, since

$$\mathcal{A}(\Gamma^m x_m) = (\Gamma^m x_m)(d-1-\mathcal{A}) \, ,$$

as can be checked easily by multiplying the gamma matrices and using obvious symmetries. Thus we can put

$$\Omega^{(-)}_{l'M} = \Gamma^m \hat{x}_m \, \Omega^{(+)}_{lM} \, ,$$

where $(d-2+l') = (d-1+l)$, thus $l' = l+1$ as for $d = 2, 3$. Conversely we have

$$\Gamma^m \hat{x}_m \, \Omega^{(-)}_{l'M} = \Omega^{(+)}_{lM} \, .$$

Another multiplication of gamma matrices yields the Dirac operator in the form

$$i\Gamma^m p_m = \Gamma^m \hat{x}_m \frac{d}{dr} + \frac{1}{r} \Gamma^m \hat{x}_m \, \mathcal{A} \, .$$

With

$$\psi = f_1(r) \, \Omega^{(-)}_{l+1,M} \; + \; f_2(r) \, \Omega^{(+)}_{lM}$$

the Dirac equation takes the form

$$f_1' - \frac{l}{r} f_1 + \sigma_c f_2 = 0 \qquad (3.9a)$$

$$f_2' + \frac{l+d-1}{r} f_2 + \sigma_c f_1 = 0 \, . \qquad (3.9b)$$

We see that both (3.2) and (3.8) for $d = 2$ and $d = 3$ are of the same form as (3.9) for arbitrary d.

3.4 Nonexistence

The presentation here is adapted to both sigma model as well as to Goldstone model backgrounds, namely for both 'inatanton', (2.13), and 'soliton', (2.14), profiles of $\sigma_c(r)$. Let us analyse the $d = 2$ equation (3.2), noting that the same conclusions hold for the arbitrary case (3.9).

In the $r \ll 1$ region, the solutions of equations (3.2a,3.2b) which are differentiable at the origin, have the asymptotic forms

$$\begin{aligned} f_1 &\approx Ar^l \left(1 + \frac{\sigma_c(0)^2}{4(l+1)} r^2\right) \\ f_2 &\approx -\frac{A\sigma_c(0)}{2(l+1)} r^{l+1} \left(1 + \frac{\sigma_c(0)^2}{4(l+2)} r^2\right). \end{aligned}$$

In particular, $f_2(0) = 0$ and $f_1(0)$ is finite for all possible values of l. At infinity, both f_1 and f_2 decay exponentially.

Now the equations (3.2) yield

$$-f_1 f_1' + f_2 f_2' + \frac{1}{r}\left[l f_1^2 + (l+1) f_2^2\right] = 0\,.$$

Integrating over r and using the said boundary conditions, one finds

$$\frac{1}{2}\, f_1(0)^2 + \int_0^\infty \frac{1}{r}\left[l f_1^2 + (l+1) f_2^2\right] dr = 0\,,$$

which clearly is impossible. There are therefore no solutions of (2.11) satisfying (2.12) in both the backrounds (2.13) and (2.14), and in any d.

4. Type (ii) models

Type (ii) models with action (2.2) separate in two main subclasses, namely those with $\mu = 0$, presented in the first subsection 4.1, and those with $\mu > 0$, presented in the second subsection 4.2. The quantity $\Xi[\varphi(x_m)]$, defined in section 2.2, will be specified further in two cases. The first of these is a particular model with codimension-$d = 2$ leading to the recovery of the result of [8, 9], presented in subsection 4.2.1, while the second is for the generic codimension-$d \geq 2$ models, presented in subsection 4.2.2.

$\mu = 0$ models of type (ii) are defined exclusively in even codimension-$d (= 2n)$, since YM 'instanton' backgrounds exist only in even dimensions.[3] $\mu > 0$ models of type (ii) are defined for all codimension-d, in Higgs or Goldstone 'soliton' backgrounds. In the latter case we shall eschew the Higgs backgrounds in favour of the corresponding *associated* Goldstone backgrounds in which the gauge field is suppressed.

[3]It is also possible for grassmannian sigma models, in which case the Dirac operator D_m in (2.2) features a *composite connection* in terms of the grassmannian field. Once we know this is a possibility, it is superfluous to present it in detail here.

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4.1 Type (ii) models with $\mu = 0$: even $d \geq 4$ isospinor $\psi(x_m)$

Here, the solitons in whose background the residual Dirac equation (2.11) is to be solved are restricted to the 'instantons' of $d = 2n$ dimensional YM models described in appendix A. To solve equation (2.11) with $\mu = 0$ in the spherically symmetric background for the YM connection (2.21), we subject the isospinor $\psi(x_m)$ to spherical symmetry

$$\psi = f_1(r)\,\mathbb{1} + f_2(r)\,\Sigma_m^{(\pm)}\,\hat{x}_m\,. \tag{4.1}$$

Substituting (4.1) and the radial Ansatz (2.21) for the YM connection into the residual Dirac equation (2.11), with $\mu = 0$, we find the familiar [14] solutions

$$f_1 = r^{-\frac{1}{2}(d-1)} \mathrm{e}^{\frac{1}{2}(d-1)\int \frac{w}{r}\,dr} \tag{4.2a}$$

$$f_2 = r^{-\frac{1}{2}(d-1)} \mathrm{e}^{-\frac{1}{2}(d-1)\int \frac{w}{r}\,dr}\,. \tag{4.2b}$$

Adopting the asymptotics (2.22), it is easy to see that $f_1(r)$ satisfies the required condition (2.12), while $f_2(r)$ does not and must be rejected[4]. It also follows from (2.22) that the relevant solution, e.g. $f_1(r)$ has a *power decay* at infinity.

4.2 Type (ii) models with $\mu \neq 0$

In this case, the separability Ansatz (2.7), and the resulting Yukawa terms like (2.31), will be specified in the two distinct cases of a particular $d = 2$ model featuring a column-valued Dirac spinor on the codimension, and the generic $d \geq 3$ models with SO(d) isospinor Dirac fields, which form square matrix arrays. These cases will be presented in the following two subsections, 4.2.1 and 4.2.2.

In both 4.2.1 and 4.2.2, the essential procedure is to so select the separability Ansatz (2.7), such that the resulting residual Dirac equation (2.11) turns out to be a known problem leading to normaisable zero modes satisfying (2.12).

The analysis of the residual Dirac equation (2.11) for these models will be restricted to 'soliton' backgrounds of the *associated Goldstone* models, cf. appendix C, rather than the corresponding backgrounds of the Higgs models, cf. appendix B. The former are the gauge decoupled versions of the latter and the results of these analyses are qualitatively the same. Accordingly, (2.11) is effectively replaced in what follows by

$$(\Gamma^m \partial_m + \mu\,\xi_c)\,\psi = 0\,. \tag{4.3}$$

i.e. with ∂_m replacing the D_m in (2.11).

4.2.1 Type (ii) model with $\mu \neq 0$: $d = 2$

For $d = 2$ the use of a doublet (ϕ^1, ϕ^2) with a scalar and a pseudoscalar component yields an interesting special situation. We consider $U(1)$ gauge fields, such that in this model $\psi(x_m)$ does not carry an isotopic index. Specifying the separability Ansatz (2.7) with (2.32), namely by

$$\xi = \sigma_1 \sigma_m \phi^m = \phi^1 + i\sigma_3 \phi^2\,,$$

[4]When instead the alternative asymptotics pointed out in footnote 2 is adopted, then $f_2(r)$ satisfies (2.12) and it is $f_1(r)$ that must be rejected.

we end up essentially with the model of [8, 11]. The most interesting feature of this model is the presence of fermionic zero modes for abelian backgrounds with any vorticity n.

With (2.32) in the residual Dirac equation (2.11), one proceeds to solve the latter in the background of the $d = 2$ Higgs 'soliton', namely the usual Nielsen-Oleson vortex or another ($p \geq 2$) member of the hierarchy in appendix B, e.g. the vortex of the system (B.8).

Alternatively, as in effect we will, one can solve (2.11) in the background of the $d = 2$ Goldstone 'soliton' of the $p = 2$ member of the hierarchy in appendix C, namely the vortex of the system (C.3), i.e. the one resulting from the gauge decoupling of the system (B.8). We restrict the subsequent analysis to that of (2.11) in the associated Goldstone 'soliton' background.

Substituting (2.32) with ϕ^m given by (2.27) and the radially symmetric Ansatz (3.1) for ψ, in (2.11) for $d = 2$, the latter separates for $m' = m + 1$, and reduces to the pair of coupled first order equations

$$f_1' - \frac{m}{r} f_1 + \eta\, h\, f_2 = 0 \tag{4.4a}$$

$$f_2' + \frac{n+m+1}{r} f_2 + \eta\, h\, f_1 = 0\,. \tag{4.4b}$$

The Dirac equations in [11, 8, 10] reduce to Eqns. (4.4a),(4.4b), reproducing the $d = 2$ result of [8, 10], for completeness.

4.2.2 Type (ii) models with $\mu \neq 0$: $d \geq 2$ isospinor $\psi(x_m)$

The situation here is similar to the case of 'instanton' backgrounds considered in 4.1 and likewise the isospinor Dirac field subject to spherical symmetry is

$$\psi = f_1(r)\, \mathbb{1} + f_2(r)\, \Gamma_m \hat{x}_m\,. \tag{4.5}$$

Note here that in (4.5) we have Γ_m in all d dimensions, while in (4.1) we have chiral matrices $\Sigma_m^{(\pm)}$ in $d = 2n$, even, dimensions.

The separability Ansatz in these cases is specified by (2.33), namely

$$\xi = \mathbb{1} \otimes \Gamma_m \phi^m\,.$$

The residual Dirac equation (4.3) in the Goldstone 'soliton' background now separates[5] and yields the following pair of first order equations

$$f_1' + \eta h\, f_1 = 0 \tag{4.6a}$$

$$f_2' + \frac{d-1}{r} f_2 + \eta h\, f_2 = 0\,, \tag{4.6b}$$

leading to

$$f_1 = e^{-\eta \int h\, dr} \tag{4.7a}$$

$$f_2 = \frac{1}{r^{d-1}} e^{-\eta \int h\, dr}\,. \tag{4.7b}$$

[5]Note here that for the case $d = 2$, where the abelian Higgs (or Goldstone) background (2.27) is radial for all vorticity n, this separation can take place only for the *unit* vorticity $n = 1$ background. This contrasts with the model of [11, 8] presented above in 4.2.1.

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Given the asymptotics of $h(r)$, (2.28), and the behaviour of $h(r)$ near the origin to be

$$h(r) \approx b\,r\,, \tag{4.8}$$

it follws that both f_1 and f_2 vanish asymptotically in the $r \gg 1$ region as required, but only f_1 converges in the $r \ll 1$ region while f_2 diverges and must be rejected. The result is one normalisable zero mode, $f_1(r)$. It follows from (2.28) that the solution $f_1(r)$ has an *exponential decay* at infinity.

The corresponding result, for (2.11) in the Higgs 'soliton' background, can be readily found using the Ansatz (2.21), with the matrices $\Sigma_m^{(\pm)}$ there replaced by Γ_m, *viz.* (B.4) in appedix B. The result is qualitatively the same, with the zero mode $f_1(r)$ still localised exponentially, except that the energy density of the YMH background brane is power localised rather than that of the Goldstone background brane analysed here, which is exponentially localised.

Instead of one such zero mode, it is possible to construct a family of such solutions by relaxing the constraint of spherical symmetry. In particular, imposing only axial symmetry characterised by a vortex number n, a family of such solutions labeled by n can be found. We do not present the details here.

5. Summary

We have addressed the problem of extending the mechanism of confining a fermion to the brane in a $4+1$ dimensional model, proposed in [2], to the case of $4+d$ dimensional models, for arbitrary d.

In the model of [2, 3], the confinement mechanism relies on the fact that a scalar field model in the 1 dimensional extra coordinate, i.e. the codimension-1, supports topologically stable 'soliton' solutions. This scalar field enters the 5 dimensional fermionic model through a Yukawa interaction term and results in the Dirac equation of the system developing a mass term asymptotically in the codimension, which is responsible for the confinement.

To extend this mechanism to higher dimensions, it seems [3] natural to employ some field theoretic model on the codimension-$d > 1$ that supports topologically stable finite energy solutions. We considered candidates for such models which support either 'soliton' or 'instanton' like solutions. Our nomenclature throughout was that 'solitons' are supported by Higgs or Goldstone models, most notably featuring dimensionful scalar fields whose self-interaction potential leads to symmetry breaking. 'Instantons' on the other hand are supported by purely YM models or by sigma models, in even dimensions.

We proposed two types of models, in both of which the separation of the Minkowski space coordinates x_μ from the codimension-d coordinates x_m was effected by an Ansatz, which also resulted in the dimensional descent of the Dirac equation in $4+d$ dimensions, to one in d dimensions, which we referred to as the *residual Dirac equation.* The solutions of the latter were what described the localisation of the fermion to the brane.

The first type, (i), of models was characterised by a Dirac operator, which featured a partial derivative in all components of the differential operator. Consequently, the residual Dirac spinors were *isoscalars.* The information on the topologically stable solutions on whose background the residual Dirac equation was solved, was encoded in a scalar coefficient in the Yukawa term. This quantity was a descendent of the topological invariant of the background system. It was found that the residual Dirac equation of these models did not support normalisable zero modes.

The second type, (ii), of models was characterised by a Dirac operator, which featured a partial derivative in the minkowskian components of the differential operator and a covariant derivative for the components on the codimension. Consequently, the residual Dirac spinors were *isospinors* for $d \geq 3$ when the gauge group was non abelian, and only in the $d = 2$ case when the gauge group was abelian it was *isoscalar*. Type (ii) models did result in normalisable zero modes for the residual Dirac equations, provided that the Yukawa term was chosen appropriately, and in the case of pure YM 'instanton' backgrounds this meant its absence. When 'instanton' backgrounds were employed localisation to the brane featured a power of r, while employing 'soliton' backgrounds of Higgs or Goldstone models resulted in exponential localisation.

A nontrivial aspect of our results is that in the models presented in sections 4.2, the Higgs models, i.e. the YMH systems, can be contracted down to the *associated* Goldstone models by elimination of the gauge fields, cf. appendix C. Whether this additional feature in constructing such models is of any physical advantage is not obvious, but we note that the energy density of the Higgs model brane system is power localised, while that in the Goldstone case is exponentially localised. Also, in some of the work of [8], a Goldstone model has been employed, albeit a model with divergent energy.

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A. Yang-Mills models in $d = 2n$ dimensions

In all even $d = 2n$ dimensions, it is possible to construct Yang-Mills (YM) models which support 'instanton' like solutions, which are topologically stable and their energy integrals are finite. The fundamental relation that ensures the existence of 'instanton' in this hierarchy [12] of YM systems are the inequalities stating the lower bounds on the energy integrals, given by the appropriate topological charges, namely the Chern-Pontryagin (C-P) numbers.

Using the convenient notation for the $2p$-form p-fold totally antisymmetrised product of the curvature 2-form $F \equiv F(2)$,

$$F(2p)_{\mu_1\mu_2\dots\mu_{2p}} = F_{[\mu_1\mu_2}F_{\mu_3\mu_3}\dots F_{\mu_{2p-1}\mu_{2p}]}\,, \quad \text{totally antisymmetrised in } [\mu_1\mu_2\dots\mu_{2p}]\,, \tag{A.1}$$

the simplest such inequality for a $2(p+q)$ dimensional YM system states

$$\mathrm{Tr}\left(|F(2q)|^2 + \kappa^{2(p-q)}\frac{(2q)!}{(2p)!}|F(2p)|^2\right) \geq 2\kappa^{p-q}\,\mathrm{Tr}\,F\wedge F\wedge\dots\wedge F\,, \quad p+q \;\; \text{times}\,, \tag{A.2}$$

where κ is a constant with the dimensions of length (if $p > q$). The left hand side defines the energy density of the YM system and the right hand side is proportinal to the $(p+q)$-th C-P density.

In the special case where $p = q$, the $4p$ dimensional YM systems are scale invariant and the inequality (A.2) can be saturated. If the gauge group is chosen to be SO($4p$) and the gauge fields are in the chiral representations thereof then instanton solutions can be evaluated explicitly in the spherically symmetric cases, but these do not interest us here.

What is relevant in the present context is the family of YM models in $4p$ dimensions from which the $d(< 4p)$ dimensional Higgs models described in appendix B below are constructed. Also relevant are the $2(p + q) = 2n$ dimensional models defined by (A.2), in any even dimension, whose solutions [27] provide the 'instanton' backgrounds used in section 4.1. Solutions to both these types of YM models, scale invariant or otherwise, are 'instantons' in the sense that at infinity the gauge connection is *pure gauge* satisfying

$$A_m \underset{r\to\infty}{\longrightarrow} g^{-1}\partial_m g\,, \tag{A.3}$$

such that in the spherically symmetric case (2.21) the 'instanton' profile (2.22) obtains.

B. Higgs models in d dimensions

In this appendix, we describe the Higgs models [18]–[20] in d dimensions which support finite energy topologically invariant soliton solutions.

In any given dimension d, a hierarchy of Higgs models supporting solitons can be systematically constructed by subjecting the p-th member of the Yang-Mills hierarchy [12], cf. appendix A, in dimension $4p > d$, to dimensional reduction down to d dimensions. The descent mechanism essentially consists of the imposition of a symmetry, which results in the breaking of the gauge group of the original $4p$ dimensional YM system, and at the same time the components of the gauge connection on the extra $4p - d$ dimensions appear as Higgs fields in the residual d dimensional system, namely in a Higgs model. By choosing the gauge group of the $4p$ dimensional system suitably, the gauge group of the $4p$ dimensional YM system breaks down to SO(d), yielding the required d dimensional SO(d) Higgs model [18]–[20]. For simplicity we will restrict to the scale invariant YM systems for the present purpose.

Now the action density of the $4p$ dimensional scale invariant YM system is bounded from below by the $2p$-th Chern-Pontryagin (C-P) density. It turns out that under the descent mechanism described in the previous paragraph, this topological lower bound translates to a new lower bound on the energy density of the residual d dimensional Higgs model. The lower bound is given by the residual C-P density, which now depends on the residual gauge group, not necessarily SO(d). Such lower bounds might be described as *Bogomol'nyi bounds*. That this residual C-P density is a topological charge density follows from the fact that it can be shown to be a total divergence [28], whose resulting surface integral turns out to be finite subject to the usual *symmetry breaking* asymptotics of the Higgs field *provided* that the residual gauge connection also exhibits the requisite asymptotics.

In the particular case of interest, namely when the residual gauge group is arranged to be SO(d), the Higgs multiplet is

$$\Phi = \Gamma_m \, \phi^m \,, \tag{B.1}$$

where the index $m = 1, 2, \ldots, d$ labels also the coordinates x_m, in the same notation as above. The *symmetry breaking* condition of the Higgs field can then be stated as

$$|\phi|^2 = \phi^m \phi^m \underset{r\to\infty}{\longrightarrow} \eta^2 \,, \tag{B.2}$$

where η is the VEV, related to the compactification scale used in the dimensional descent from $4p$ dimensions. For $d \geq 3$, i.e. when the residual gauge group is non abelian, the Higgs field points along the unit vector $\hat{x}_m$ in the $r \gg 1$ asymptotic region, i.e. on the $d-1$ sphere. In this region the gauge group breaks down to SO($d-1$) and the connection field decays as r^{-1}. In the Dirac gauge, where the Higgs field in the $r \gg 1$ asymptotic region points along the d-th direction, the connection develops a semi-infinite line singularity in the x_d direction, which is an artefact of this gauge. This analogy with the familiar case of the monopole [16] in the $d = 3$ case is complete and the residual SO(d) connection behaves as

$$A_m \underset{r\to\infty}{\longrightarrow} \frac{1}{2} g^{-1} \partial_m g \,, \tag{B.3}$$

namely as *half a pure-gauge*, rather than as one *pure-gauge* like an instanton. It is for this reason that above, we have called the finite energy topologically stable solutions of Higgs models 'solitons', in contrast with the corresponding solutions of even dimensional sigma models and YM systems, as 'instantons'.

In terms of the spehrically symmetric Ansatz, which is (2.21) with $\Sigma_{mn}^{(\pm)}$ now replaced by Γ_{mn}

$$A_m = \frac{1 - w(r)}{r} \, \Gamma_{mn} \hat{x}_n \,, \tag{B.4}$$

the asyptotics of the function $w(r)$ corresponding to (B.3) are

$$\lim_{r\to 0} w(r) = +1 \quad , \quad \lim_{r\to\infty} w(r) = 0 \,. \tag{B.5}$$

The only exception to the property (B.3) is the $d = 2$ Higgs model, in which case the boundary of the space is not sufficiently large to accommodate a Dirac gauge.

The most familiar Higgs models which can be construced in this scheme descend from the usual $SU(2)$ YM system in 4 dimensions, i.e. from the first ($p = 1$) member of the YM hierarchy [12], down to $d = 3$ and $d = 2$ respectively. In $d = 3$ one finds the SO(3) Georgi-Glashow model in the Prasad-Sommerfield limit, and in $d = 2$, the familiar abelian Higgs model. To illustrate this scheme further we have to proceed to $p \geq 2$, and for the sake of ease of presentation, we restict to the first two nontrivial examples. These are the SO(d) Higgs models arising from ($p = 2, d = 3$) and ($p = 2, d = 2$).

The $d = 3$, SO(3) Higgs model [19] is defined by the lagrangian $\mathcal{L}$ which is bounded from below by the topological charge density ϱ,

$$\mathcal{L} = \mathrm{Tr}\left(\{F_{[ij}, D_{k]}\Phi\}^2 + 6\lambda_3(\{S, F_{ij}\} + [D_i\Phi, D_j\Phi])^2\ , +27\lambda_2\{S, D_i\Phi\}^2 + 54\lambda_1 S^4\right) \tag{B.6}$$

$$\varrho = 36\varepsilon_{ijk}\partial_k\,\mathrm{Tr}\left[\phi\left(3\eta^4 - 2\eta^2\Phi^2 + \frac{3}{5}\Phi^4\right)F_{ij} - 2\eta^2\Phi D_i\Phi D_j\Phi - \frac{2}{5}\Phi^2(2\Phi D_i\Phi - D_i\Phi\Phi)D_j\Phi\right], \tag{B.7}$$

in which Φ is given by (B.1), $S = \eta^2 - \Phi^2$, and the manifestly total divergence form of ϱ is displayed in (B.7). Solutions to this system, were constructed in [19].

The $d = 2$, SO(2) or $U(1)$ Higgs model [20] is defined by the lagrangian $\mathcal{L}$ which is bounded from below by the topological charge density ϱ,

$$\mathcal{L} = \lambda_2[(\eta^2 - |\varphi|^2)F_{ij} + iD_{[i}\varphi^* D_{j]}\varphi]^2 + 24\lambda_1(\eta^2 - |\varphi|^2)^2|D_i\varphi|^2 + 18\lambda_0(\eta^2 - |\varphi|^2)^4 \tag{B.8}$$

$$\varrho = \varepsilon_{ij}\,\partial_i\left[\eta^6 A_j - 3i\left(\eta^4 - \eta^2|\varphi|^2 + \frac{1}{3}(|\varphi|^2)^2\right)\varphi D_j\varphi^*\right], \tag{B.9}$$

where we have used the complex valued Higgs field $\varphi = \phi^1 + i\phi^2$, and again the topological density ϱ is displayed in manifestly total divergence form.

It is easy to see that the leading terms making a nonvanishing contribution to the integrals of the topological charge densities (B.7) and (B.9), respectively, are the *magnetic charge* of the monopole and the *winding number* of the Nielsen-Oleson vortex.

C. Goldstone models associated to the Higgs models in d dimensions

In this appendix we define what we have referred to in the above as the Goldstone models *associated* [21, 22, 23] to the Higgs models in d dimensions described in appendix B above, which also support soliton solutions.

The aspect of (B.6) and (B.8) that concerns us here is that both possess gauge decoupling limits, which is a consequence of the fact that when the gauge fields are removed [21] from these densities, the remaining density still satisfies the Derrick scaling requirement. Indeed, in [20] this gauge decoupling was demonstrated concretely for the numerically constructed solutions: specifically, the Goldstone 'soliton' in [23] is obtained by gauge decoupling the 'soliton' in [20]. This feature is in stark contrast with the same procedure for the *usual* $d = 3$ and $d = 2$ Higgs models arrived at from $p = 1$ YM, i.e. the Georgi-Glashow model and the abelian Higgs model, in which cases the solitons do not persist after gauge decoupling.

It is thus possible [21] to find Goldstone models associated to each d dimensional SO(d) Higgs model descended from all $p \geq 2$ members of the YM hierarchy. We demonstrate this prescription in the two examples considered explicitly in appendix B.

In the $(p = 2, d = 3)$ model, eliminating the gauge field in (B.6),(B.7) we find the lagrangian and topological charge density of the *associated* $(p = 2, d = 3)$ Goldstone model

$$\mathcal{L} = \mathrm{Tr}\left(6\lambda_3[\partial_i\Phi, \partial_j\Phi]^2 + 27\lambda_2\eta^4(\eta^2 - \Phi^2)^2\partial_i\Phi^2 + 54\lambda_1(\eta^2 - \Phi^2)^4\right) \tag{C.1}$$

$$\varrho = 36\varepsilon_{ijk}\partial_k\mathrm{Tr}\left[-2\eta^2\Phi\partial_i\Phi\partial_j\Phi - \frac{2}{5}\Phi^2(2\Phi\partial_i\Phi - \partial_i\Phi\Phi)\partial_j\Phi\right]. \tag{C.2}$$

Note that covariant derivatives in (B.6) are replaced by partial derivatives in (C.1). This is the rationale behind the corresponding replacement of D_m in (2.11) by ∂_m in (4.3).

The system described by (C.1) is related to that considered in [22], except that in the latter, we have selected specific values of the couplings $\lambda_{(i)}$, and also replaced the symmetry breaking potential in (C.1) by other postitive definite (and symmetry breaking) potentials, without compromising the existence of the solutions. Note that in the asymptotic region, (C.2) is equivalent to the *winding number* density.

In the $(p = 2, d = 2)$ model, eliminating the gauge field in (B.8),(B.9) we find the lagrangian and topological charge density of the *associated* $(p = 2, d = 2)$ Goldstone model

$$\mathcal{L} = \lambda_2|\partial_{[i}\varphi^*\partial_{j]}\varphi|^2 + 24\lambda_1(\eta^2 - |\varphi|^2)^2|\partial_i\varphi|^2 + 18\lambda_0(\eta^2 - |\varphi|^2)^4 \tag{C.3}$$

$$\varrho = -3i\varepsilon_{ij}\,\partial_j\left[\left(\eta^4 - \eta^2|\varphi|^2 + \frac{1}{3}(|\varphi|^2)^2\right)\varphi\partial_j\varphi^*\right], \tag{C.4}$$

which is the system investigated in [23], except that in the latter two specific symmetry breaking potentials in addition to that in (C.3) were employed in the numerical construction of the solutions.

The above described procedure of constructing the *associated* Goldstone model for any SO(d) Higgs model characterised by any (p, d), can be carried out.

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